Qubit Recycling Revisited

Analysis, Generalization, and Verification of a Quantum Circuit Transformation

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Background

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- Quantum *circuits* pass qubits through *gates* that manipulate their state

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$$circuit\ scale \propto implementation\ difficulty$$
$$\Downarrow$$
$$smaller\ scale = better?$$

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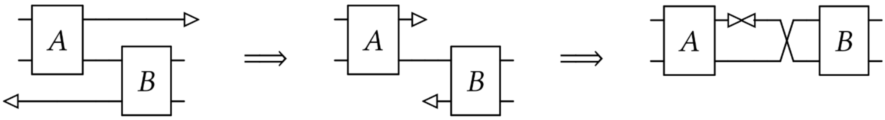
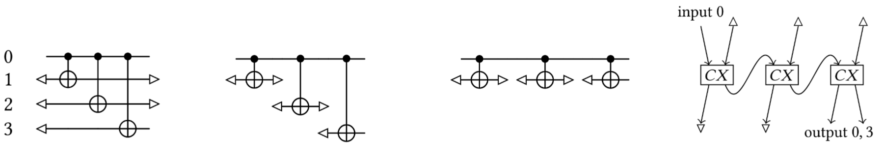
Let's talk recycling.

Recycling at a High Level





(a) Input circuit. (b) Topological deformation. (c) Renaming and reusing. (d) DAG repr. of Fig. 1a.

Fig. 1. A running example of qubit recycling.

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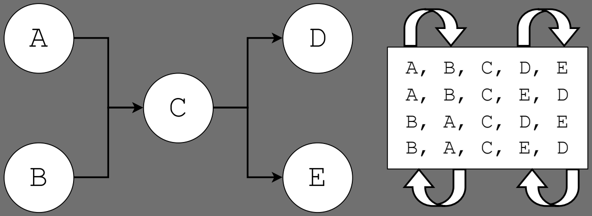
- #P (Sharp-P) is the class of problems that involves counting the number of ways to solve an NP-class problem. A problem is #P-complete if it is as hard as the hardest problem in #P



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- Every qubit recycling solution has a corresponding recycling strategy
- Idea: search for the largest recycling strategy (has most qubit reuses) instead of searching for smallest topological ordering
- The set of valid recycling strategies is much smaller than the set of topological orderings, so this is much more attainable!

Valid Recycling Strategies

Because recycling strategies are just relationships between the qubits of a circuit, we should be able to enumerate all strategies and then check for the valid ones.

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Qubit Dependency Graph (QDG) - a directed graph with qubits as vertices. $a \to b$ denotes that b is *computationally dependent* on the value of a:

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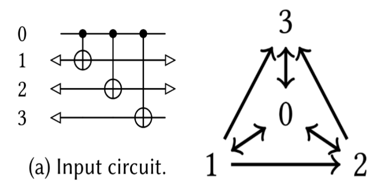
This makes sense - a cycle in $\to\hookrightarrow$ means that our recycling strategy has a circular dependency - not allowed!

Valid Recycling Strategies



Example circuit and corresponding QDG

$\{1 \hookrightarrow 2, 2 \hookrightarrow 3\}$ is a valid recycling strategy because[1]

$$\neg \exists\ a\ b,\ \text{s.t. } a \to b \hookrightarrow a$$

While $\{2 \hookrightarrow 1\}$ is not a valid strategy because of the existence of the cycle

$$1 \to 2 \hookrightarrow 1$$

[1] $a \to b$ here could generate a cycle via $2 \to 0 \to 1$, but $\to\hookrightarrow$ is not acyclic because 0 isn't reusing 2

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- Can we even find the largest strategy efficiently?
- If we can't, can we at least find a pretty large one?

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So, if an adjacency matrix is nilpotent, it cannot be cyclic because there is a maximum number of steps k between any two nodes in the graph.

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Now, we know that[2]

$$AR \text{ is nilpotent} \Leftrightarrow \exists P, P^T A(RP) \text{ is strictly lower triangular}$$

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It can be shown that Wilf's problem reduces to the strategy maximization[3], therefore qubit recycling strategy maximization is proven to be $\boxed{\text{NP-hard}}$!

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[3] Not important how, TL;DR is that it involves padding a matrix until it becomes a valid QDG of some circuit

Solving NP-hard Strategy Maximization

Now that we know how hard the problem is, we begin to solve the strategy maximization problem in a reasonable amount of time using known techniques for Wilf's problem:

	0	1	2	3
0	1	1	1	1
1	1	1	0	1
2	1	0	1	1
3	1	0	0	1

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Core idea: shift zeros to the top right corner (make matrix lower triangular). The indices of the diagonal elements of the upper-right-side submatrix correspond with a highly-optimal rewriting solution!

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Core idea: shift zeros to the top right corner (make matrix lower triangular). The indices of the diagonal elements of the upper-right-side submatrix correspond with a highly-optimal rewriting solution! Above, the indices are $(2, 3), (1, 2)$ - that's the same as the previously-mentioned strategy $\{1 \hookrightarrow 2, 2 \hookrightarrow 3\}$! Wow!

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Recap

So far, we've seen that:

- It's too difficult to search for topologically-identical but smaller quantum circuits
- It's much easier to search for valid *recycling strategies*
- Finding the largest recycling strategy is NP-hard, proven via a reduction from Wilf's matrix triangularization problem
- Diagonals on the upper-right-side submatrix of a lower-triangular adjacency matrix for a given QDG correspond with highly-optimal recycling strategies

Putting it All Together

A theoretical framework is great, but what we really want is an optimizing compiler to do all of the work for us:

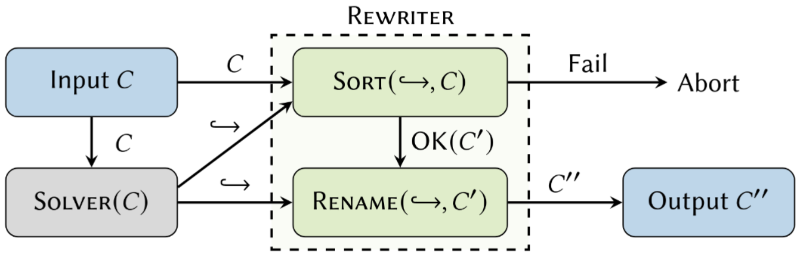

Fig. 5. Structure of the certified qubit recycler.

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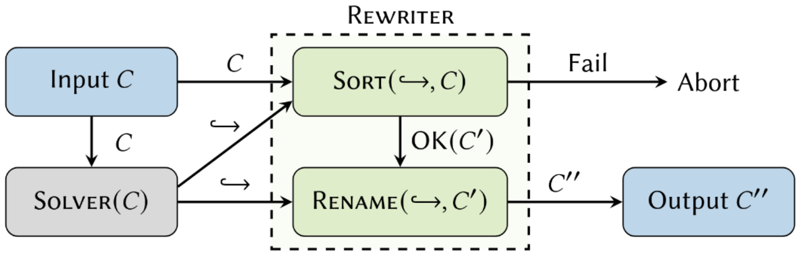

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This compiler calls the previously-described **Solver** to generate rewriting strategies, then uses its **Rewriter** to either fail if the generated strategy is invalid, or produce a *semantically-equivalent* circuit C''.

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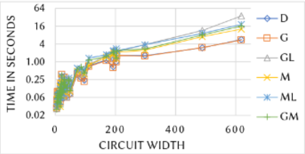
- Because the topological sort is proven correct, we know that it does not alter the semantics of the original circuit C
- The **Rename** module connects together the topologically-deformed circuit C' according to the provided strategy $\hookrightarrow$, also verified to be semantic-preserving

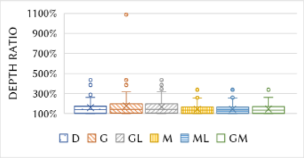
Experimental Results

Circuit	W	#Recycled qubits (the more the better) P	D	G	GL	M	ML	GM
pdc_307	619	464	505	505	505	508	508	508
spla_315	489	401	407	407	407	407	407	407
hwb9_304	170	81	121	121	119	119	119	121
ex5p_296	206	107	127	127	127	125	125	127
e64-bdd_295	195	114	126	126	126	126	126	126
hwb8_303	112	52	73	73	73	73	73	73
hwb7_302	73	31	45	45	45	44	44	45
hwb6_301	46	20	22	22	23	22	22	22

(a) For each circuit, we list its width in column "W", and the number of recycled qubits using various methods in sub-columns of "# Recycled qubits". Each sub-column corresponds to a method as follows: "P": those reported in [Paler et al. 2016]; "D": our implementation of [DeCross et al. 2023]'s algorithm; "G": Greedy; "M": Max0s; "GL": Greedy+LA; "ML": Max0s+LA; "GM": Greedy+Max0s. The best results among the methods are highlighted.



(b) Average time consumption.



(c) Box plot of the ratios of circuit depths after and before recycling.

Takeaways

Paper DOI

- *Qubit Recycling* aims to reduce the number of qubits used in a circuit
- Existing Qubit Recycling strategies both do not always provide optimal solutions and are not guaranteed to maintain semantics (behavior) of a quantum circuit
- Jiang introduces *Qubit Dependency Graphs* as a key generalization, allowing for verifiable and usually-optimal recycling solutions
- Recycler algorithm is formally verified in the Coq Proof Assistant, showing that it always maintains the semantics of rewritten circuits